

# Spin oscillations of relativistic fermions in the field of a traveling circularly polarized electromagnetic wave and a constant magnetic field

B. V. Gisin

*IPO, Ha-Tannaim St. 9, Tel-Aviv 69209, Israel. E-mail: borisg2011@bezeqint.net*

(Dated:)

The Dirac equation, in the field of a traveling circularly polarized electromagnetic wave and a constant magnetic field, has singular solutions, corresponding the expansion of energy in vicinity of some singular point. These solutions described relativistic fermions. States relating to these solutions are not stationary. The temporal change of average energy, momentum and spin for single and mixed states is studied in the paper. A distinctive feature of the states is the disappearance of the longitudinal component of the average spin. Another feature is the equivalence of the condition of fermion minimal energy and the classical condition of the magnetic resonance.

Finding such solutions assumes the use of a transformation for rotating and co-moving frames of references. Comparison studies of solutions obtained with the Galilean and non-Galilean transformation shown that some parameters of the non-Galilean transformation may be measured in high-energy physics.

PACS numbers: 03.65.Ge, 71.70.Di, 13.49.Em;

## INTRODUCTION

Recently, a new class of exact localized non-stationary solutions of the Dirac equation in the field of a traveling circularly polarized electromagnetic wave and a constant magnetic field was presented [1]. These solutions correspond to stationary states (Landau levels [2]) in a rotating and co-moving frame of references. In contrast to classical case, the wave function of such solutions never can be presented as large and small two-component spinor. They are relevant only for relativistic fermions.

Solutions, corresponding to the expansion of energy in vicinity of some singular point, or singular solutions, are of interest from the experimental viewpoint. Since solutions are non-stationary, particular attention in the paper is given to the study of the temporal behavior of the average energy, momentum and spin.

In the paper the Galilean transformation is used for the transition to the rotating and co-moving frame of references, the term "rotating frame" relates below to the given case. The term "initial frame" relates to the resting (laboratory) frame.

But the issue arises what should be the transformation Galilean or not? It means, is time the same in the rotating frame and the initial?

Some theoretical and experimental aspects of the problem in application to optics are presented in [3]. Simple two-dimensional transformation  $\tilde{\varphi} = \varphi - \Omega t$ ,  $\tilde{t} = -\tau\varphi + t$  was discussed in this paper as an example. Existing experimental data in optics [4] allow to determine that the upper boundary of the parameter  $\tau$  is  $\sim 10^{-23}$  sec [3]. In terms of length this corresponds to a distance of the order of the proton size.

Unfortunately, the parameter  $\tau$  vanishes from final results, for bounded and square integrable solutions of Dirac's equation in the case the two-dimensional trans-

formation, after the mandatory reverse transformation to the initial frame.

Comparison studies of solutions corresponding to a general form of the three-dimensional non-Galilean transformation is performed in the paper. This transformation not only is considered as a convenient tool for finding solutions, but also as having a more general meaning.

## SOLUTIONS OF DIRAC'S EQUATION

We consider Dirac's equation

$$i\hbar\frac{\partial}{\partial t}\Psi = c\boldsymbol{\alpha}(\mathbf{p} - \frac{e}{c}\mathbf{A})\Psi + \beta mc^2\Psi = 0 \quad (1)$$

in the electromagnetic field with potential

$$\begin{aligned} A_1 &= -\frac{1}{2}H_3y + \frac{1}{k}H\cos(\Omega t - kz), \\ A_2 &= \frac{1}{2}H_3x + \frac{1}{k}H\sin(\Omega t - kz), \end{aligned}$$

where  $k = \varepsilon\Omega/c$  is the propagation constant,  $\Omega$  is the frequency, the sign change of  $\Omega$  corresponds to the opposite polarization, values  $\varepsilon = 1$  and  $\varepsilon = -1$  are used when the wave propagates along the  $z$ -axis and opposite direction, respectively,  $c$  is the speed of light,  $\alpha_k, \beta$  are Dirac's matrices,  $H$  is the amplitude of this wave. This potential corresponds to a traveling plane circularly polarized wave and constant magnetic field.

By symmetry, the consideration of solutions is more convenient in a rotating frame with coordinates

$$\tilde{x} = r\cos\tilde{\varphi}, \quad \tilde{y} = r\sin\tilde{\varphi}, \quad (2)$$

$$\tilde{\varphi} = \varphi - \Omega t + kz, \quad \tilde{t} = t, \quad \tilde{z} = z, \quad (3)$$

where the tilde corresponds to this frame.

Dirac's equation (1) in the frame has exact localized stationary solutions

$$\tilde{\Psi} = \exp[-i\frac{\tilde{E}\tilde{t}}{\hbar} + i\frac{\tilde{p}\tilde{z}}{\hbar} - \frac{1}{2}d(\tilde{x}^2 + \tilde{y}^2) + d_1\tilde{x} + d_2\tilde{y}]\psi, \quad (4)$$

were  $\tilde{E}$  and  $\tilde{p}$  is the "energy" and "momentum" along the  $z$ -axis",  $\psi$  is a spinor. The equation with such a wave function may be interpreted as a two-dimensional harmonic oscillator. The wave function in the initial frame is  $\Psi = \exp[-\frac{1}{2}\alpha_1\alpha_2(\Omega t - kz)]\tilde{\Psi}$ .

For all states parameters  $d$  and  $d_1, d_2$  are determined with help of following relations

$$d = \pm \frac{eH_3}{2\hbar c} > 0, \quad d_1 = -id_2, \quad d_2 = \frac{eHcd/k}{(\tilde{E} - c\tilde{p}\varepsilon)\Omega - 2\hbar dc}, \quad (5)$$

$d$  defines the characteristic size of the localization  $l_c = \sqrt{|2\hbar c/eH_3|}$ . For simplicity, we consider below the case  $eH_3 < 0$ .

States may be classified in accordance with the form of the spinor  $\psi$ . A constant spinor describes "ground state". A spinor polynomial in  $\tilde{x}, \tilde{y}$  corresponds to "excited states".

### Ground state

The solution for the ground state  $\psi_g$  and the characteristic equation, for energy eigenvalues  $\mathcal{E}$ , in normalized units are as follows

$$\psi_g = N_g\psi_0, \quad \psi_0 = \begin{pmatrix} h\mathcal{E} \\ -\varepsilon(\mathcal{E} + 1)(\mathcal{E} - \mathcal{E}_0) \\ \varepsilon h\mathcal{E} \\ -(\mathcal{E} - 1)(\mathcal{E} - \mathcal{E}_0) \end{pmatrix}, \quad (6)$$

$$\Lambda = \frac{2\varepsilon\tilde{p}c - \hbar\Omega}{mc^2}, \quad \mathcal{E}(\mathcal{E} + \Lambda) - 1 - \frac{\mathcal{E}h^2}{\mathcal{E} - \mathcal{E}_0} = 0, \quad (7)$$

where normalized units are

$$\mathcal{E} \equiv \frac{(\tilde{E} - \varepsilon\tilde{p}c)}{mc^2}, \quad \mathcal{E}_0 = \frac{2d\hbar}{\Omega m}, \quad h = \frac{e}{kmc^2}H, \quad (8)$$

It is easy to prove the equality

$$\int \tilde{\Psi}^* \tilde{\Psi} d\tilde{x} d\tilde{y} \equiv \int \Psi^* \Psi dx dy = 1, \quad (9)$$

for the rotating frame and the initial. The normalized constant  $N_g$ , as usually, is defined from this equality.

For the  $d = +eH_3/2\hbar c$ , the wave function is  $\psi_+ = \varepsilon\alpha_1\alpha_3\beta\psi$  with the simultaneous sign change of  $\mathcal{E}_0$ .

### Excite states

For the first "excite state"  $\psi = \psi_0 + \tilde{x}\psi_x + \tilde{y}\psi_y$ , where  $\psi_0, \psi_x, \psi_y$  are constant spinors. Two types of such solutions are possible.

The first type is realized at  $\psi_y = -i\psi_x$ . The solution and the parameter  $\Lambda$  of the characteristic equation are

$$\psi_{e1} = N_{e1}\psi_0(1 - i\frac{d}{d_2}\tilde{x} - \frac{d}{d_2}\tilde{y}), \quad \Lambda = \frac{2\varepsilon\tilde{p}c - 3\hbar\Omega}{mc^2}, \quad (10)$$

where  $\psi_0$  is the same as in (6).

For the second type  $\psi_y = i\psi_x$ . The solution is

$$\psi = N_{e2} \begin{pmatrix} (\mathcal{E}\mathcal{E}_0 + 1)(\mathcal{E} - \mathcal{E}_0) - \varepsilon h k \mathcal{E} \mathcal{E}_0 (i\tilde{x} - \tilde{y}) \\ -\varepsilon(\mathcal{E} + 1)\mathcal{E}_0 [h - \varepsilon k(\mathcal{E} - \mathcal{E}_0)(i\tilde{x} - \tilde{y})] \\ \varepsilon(\mathcal{E}\mathcal{E}_0 + 1)(\mathcal{E} - \mathcal{E}_0) - h k \mathcal{E} \mathcal{E}_0 (i\tilde{x} - \tilde{y}) \\ -(\mathcal{E} - 1)\mathcal{E}_0 [h - \varepsilon k(\mathcal{E} - \mathcal{E}_0)(i\tilde{x} - \tilde{y})] \end{pmatrix}. \quad (11)$$

The eigenvalue equation differs from (7) by the parameter  $\Lambda = (2\varepsilon\tilde{p}c + \hbar\Omega)/mc^2$  and the additional term  $2\hbar\Omega\mathcal{E}_0/mc^2$

$$\mathcal{E}(\mathcal{E} + \Lambda) - 1 - \frac{2\hbar\Omega}{mc^2}\mathcal{E}_0 - \frac{\mathcal{E}h^2}{\mathcal{E} - \mathcal{E}_0} = 0. \quad (12)$$

Obviously, wave functions (6), (11) cannot be presented as a small and large two-component spinor. It means that the difference  $E^2 - m^2c^2$  cannot be small and these solutions correspond only to the relativistic case.

### SINGULAR SOLUTIONS

The above non-stationary states can be determined with help of the "energy  $\mathcal{E}$ " in the rotating frame. However, in the initial frame, we should use the average values of the operators. The same is valid for momentum and spin. From the experimental point of view, of particular interest are special solutions with energies close to the singular point  $\mathcal{E}_0$ . Such singular states for the Galilean transformation arise only by a fixed momentum and relate as to the ground as excited states.

It is well known that the eigenvalue equation, as an equation of the third order, has exact solutions  $\mathcal{E}(\mathcal{E}_0, h, \tilde{p})$ . However, for singular solutions it is more convenient to use roots of the eigenvalue equation in the form of the expansion in power series in  $h$  in a vicinity of  $\mathcal{E}_0$ . The equation for any state has a pair of such roots. Notice that the parameter  $h$  is always very small,  $h = eH/kmc^2 \ll 1$ .

Broadly speaking, parameter  $\tilde{p}$  also may depend on  $h$ . In this case the eigenvalue equation would bind coefficients of this expansion  $\mathcal{E}_k$  and  $\tilde{p}_k$  and one from these coefficients, except  $\mathcal{E}_0$  and  $\tilde{p}_0$ , remains indefinite. The possibility to define both the coefficients is discussed in the last Section. Here we assume that  $\tilde{p}$  do not depend on  $h$ .

For the ground state

$$\mathcal{E}(\mathcal{E} + \Lambda) - 1 - \frac{\mathcal{E}h^2}{\mathcal{E} - \mathcal{E}_0} = 0, \quad \Lambda = \frac{2\varepsilon\tilde{p}c - \hbar\Omega}{mc^2},$$

$$\mathcal{E} = \mathcal{E}_0 \pm \frac{\mathcal{E}_0 h}{\sqrt{\mathcal{E}_0^2 + 1}} + \frac{\mathcal{E}_0 h^2}{2(\mathcal{E}_0^2 + 1)^2} + \dots \quad (13)$$

A necessary condition of such an expansion is a certain momentum. This momentum for the ground state is

$$\tilde{p} = \frac{\varepsilon mc}{2\mathcal{E}_0} - \frac{\varepsilon mc}{2}\mathcal{E}_0 + \varepsilon \frac{\hbar\Omega}{2c}. \quad (14)$$

For the exited state of the first type the expansion coincides with (13), but the term  $\varepsilon\hbar\Omega/c$  must be added to momentum.

For the exited state of the second type this expansion is

$$\mathcal{E} = \mathcal{E}_0 \pm \frac{\mathcal{E}_0 h}{\sqrt{\mathcal{E}_0^2 + 1 + \varsigma}} + \frac{\mathcal{E}_0(1 + \varsigma)h^2}{2(\mathcal{E}_0^2 + 1 + \varsigma)} + \dots, \quad (15)$$

where  $\varsigma = 2\hbar\Omega\mathcal{E}_0/mc^2 \ll 1$ . The momentum is the same as for the ground state (14).

### Average values of operators for single states

Below we consider average energy, momentum and spin, defined as the integral of corresponding operators over all the cross-section. This integral has exact value, however, for simplicity, we use first approximation, neglecting terms with  $h$  and of the order of  $\hbar\Omega \ll mc^2$ . In this approximation the average value of energy coincides for all considered states

$$E_a \approx \frac{(\mathcal{E}_0^2 + 1)}{\mathcal{E}_0} mc^2. \quad (16)$$

For different states this value differs by terms of the order  $\hbar\Omega$ . In particular, these terms are  $+\hbar\Omega/2$ ,  $-\hbar\Omega/2$ ,  $3\hbar\Omega/2$  for the ground, first and second excited states, respectively.

A minimum of  $E_a = 2mc$  is realized at  $\mathcal{E}_0 = 1$ . Noteworthy the astonishing thing. It may be shown with help of (8), that this equality is the classical condition of the magnetic resonance

$$\mu H_z = \frac{1}{2}\hbar\Omega, \quad (17)$$

where  $\mu = e\hbar/2mc$  is the electron magnetic moment when  $g$ -factor equals two. In the general case the inverse of  $\mathcal{E}_0$  defines the  $g$ -factor.

The existence of such a minimum allows us to assume that the states have chances to be stable.

Average components of momenta equal in the first approximation

$$p_{1a} = \mp \frac{1}{2}mc\sqrt{\mathcal{E}_0^2 + 1} \cos(\Omega t - kz), \quad (18)$$

$$p_{2a} = \mp \frac{1}{2}mc\sqrt{\mathcal{E}_0^2 + 1} \sin(\Omega t - kz), \quad (19)$$

$$p_{3a} = \frac{\varepsilon mc}{\mathcal{E}_0}. \quad (20)$$

Large values of energy and components of average momenta once again can explain why the term "relativistic" is used for considered solutions.

The average spin is defined by the integral

$$s_k = \frac{\hbar}{2} \int \Psi^* \sigma_k \Psi dx dy, \quad (21)$$

where  $\sigma_1 = -i\alpha_2\alpha_3$ ,  $\sigma_2 = -i\alpha_3\alpha_1$ ,  $\sigma_3 = -i\alpha_1\alpha_2$  and the integration is over all cross-section.

A distinctive feature of singular solutions is the vanishing of  $s_3$  in the first approximation. For single states considered here the temporal behavior of transverse components are

$$s_2 = \mp \frac{\hbar}{2} \frac{\varepsilon \mathcal{E}_0}{\sqrt{\mathcal{E}_0^2 + 1}} \cos(\Omega t - kz), \quad (22)$$

$$s_2 = \mp \frac{\hbar}{2} \frac{\varepsilon \mathcal{E}_0}{\sqrt{\mathcal{E}_0^2 + 1}} \sin(\Omega t - kz), \quad (23)$$

### Mixed states

Interesting feature of the singular solutions is the existence of different states with the same momentum  $\tilde{p}$ , as, for example, the ground and exited state of the second type. It opens possibility of non-stationary, but stable mixed states. Such states may be formed only from solutions with the same sign of the first term in the expression  $\mathcal{E}$ . For different signs the integral of the average value of an operator  $P$  contains the complimentary term  $\int \Psi_m^* P \Psi_n dx dy$ , where  $m$  and  $n$  pertains to different states. This term contains a factor

$$\exp\left[-\frac{1}{2d}(d'_2 - d''_2)^2\right] \approx \exp\left[-\frac{(\mathcal{E}_0^2 + 1)}{2\lambda_e^2 d}\right], \quad (24)$$

where  $\lambda_e = 3.86 \cdot 10^{-11} cm$  is the Compton wavelength of electron. This factor is very small because  $\lambda_e^2 d \ll 1$ . In the opposite case of the same signs this factor is little differ from 1.

A mixed state consisting of the ground and exited state  $\Psi = C_g \Psi_g + C_{e2} \Psi_{e2}$ , is considered below, as an example. Indexes  $g$  and  $e2$  correspond to the ground and the excite state of the second type respectively. For simplicity, we assume that  $C_g, C_{e2}$  are real constants  $C_g = C_{e2} = 1/\sqrt{2}$ . For such a mixed state the equality of momenta is the favorable fact from the viewpoint of stability.

For this state the temporal dependence of spin components in the first approximation are  $s_3 = 0$ ,

$$s_1 = \mp \frac{\varepsilon \hbar \mathcal{E}_0}{2\sqrt{\mathcal{E}_0^2 + 1}} [1 \pm \cos \omega t] \cos(\Omega t - kz), \quad (25)$$

$$s_2 = \mp \frac{\varepsilon \hbar \mathcal{E}_0}{2\sqrt{\mathcal{E}_0^2 + 1}} [1 \pm \cos \omega t] \sin(\Omega t - kz), \quad (26)$$

$$\omega = \frac{2\mathcal{E}_0^2\omega_m}{\sqrt{(\mathcal{E}_0^2+1)^3}}, \quad \omega_m = \frac{\mu H}{\hbar}. \quad (27)$$

If  $C_g = -C_{e2} = 1/\sqrt{2}$  in (25), (26)  $\cos\omega t$  should be replaced by  $\sin\omega t$ .

Noteworthy, for comparison, the temporal dependence of the average spin in the non-relativistic case. It follows from any bounded and square integrable solutions of the Pauli equation in a rotating magnetic field and arbitrary static electric field, that this dependence is  $s_1 = (\hbar/2)\sin\omega_m t \sin\Omega t$ ,  $s_2 = (\hbar/2)\sin\omega_m t \cos\Omega t$ ,  $s_3 = (\hbar/2)\cos\omega_m t$ . In given case the average spin is determined by (21), but the integration is over all volume of the field,  $\sigma_k$  is the Pauli matrix. .

### NON-GALILEAN TRANSFORMATION FOR ROTATING FRAMES OF REFERENCE

All the above results are obtained with help of the transition to a rotating frame. For such a transition the Galilean transformation is used (2), (3). It means that time in both frames is assumed to be invariable. However, from the viewpoint of contemporary physics this assumption seems unbelievable. This is a strong argument for using a non-Galilean transformation.

In this Section a general form of normalized non-Galilean transformation

$$\tilde{r} = r, \quad \tilde{\varphi} = \varphi - \Omega t + kz, \quad (28)$$

$$\tilde{t} = -\tau\varphi + \gamma z + t, \quad \tilde{z} = \lambda\varphi + z + vt, \quad (29)$$

is considered. Here  $v$  is the velocity of fermion,  $\tau$ ,  $\lambda$  and  $\gamma$  are parameters with the dimension of time, length and inverse velocity, respectively, these parameters depend of  $\Omega$  and  $v$ .

This linear transformation corresponds to the concept of "point rotation frame" with the axis of rotation at every point [3]. This concept is used in optics more 100 years. The characteristic example of the concept is the optical indicatrix (index ellipsoid).

Normalization of  $\tilde{\varphi}$  in (28) implies that the wave function in the rotating frame is periodic with respect to  $\tilde{\varphi}$  as well as the wave function in the initial frame is periodic with respect to  $\varphi$ .

For stationary states in rotating frame the wave function differs from (4) by factor  $\exp(-iE\tilde{t}/\hbar + ip\tilde{z}/\hbar - in\tilde{\varphi})$ , which is inserted instead of  $(-i\tilde{E}\tilde{t}/\hbar + i\tilde{p}\tilde{z}/\hbar)$ . However, a necessary condition for the existence of bounded and square integrable solutions of the Dirac equation must be added

$$\tau E + \lambda p = \hbar n, \quad (30)$$

where  $n$  is a integer. States in the given case are characterized by two integers similarly to a two-dimensional harmonic oscillator. These integers are  $n$  and the degree

of polynomial  $\psi$ . This is yet another argument in favor of the non-Galilean transformation.

The condition (30) and the eigenvalue equation allows to determine both parameters  $E$  and  $p$  in the case dependence them from  $\hbar$ . This is one more argument in the favor of this transformation.

Introduce new parameters  $\tilde{E}'$  and  $\tilde{p}'$

$$\tilde{E}' = E - vp - n\hbar\Omega, \quad \tilde{p}' = -\frac{1}{v_z}E + p - \frac{\varepsilon}{c}n\hbar\Omega. \quad (31)$$

$\tilde{E}'$  and  $\tilde{p}'$  coincide with  $\tilde{E}$  and  $\tilde{p}$ , only in a first approximation. With these parameters and due to condition (30), the form of the equation for stationary states in both cases of the Galilean and non-Galilean transformation fully coincides. Using (31)  $E$  and  $p$  may be expressed in terms of  $\tilde{E}'$  and  $\tilde{p}'$  and inserted in (30).

It may be straightforwardly shown with help of the condition (30) and relations (28), (29) that in the initial frame the equality

$$(-i\frac{E\tilde{t}}{\hbar} + i\frac{p\tilde{z}}{\hbar} - in\tilde{\varphi}) = (-i\frac{\tilde{E}'\tilde{t}}{\hbar} + i\frac{\tilde{p}'\tilde{z}}{\hbar}) \quad (32)$$

is fulfilled. In this frame the shape of wave functions for both cases coincide since for the Galilean transformation  $\tilde{t} = t$ ,  $\tilde{z} = z$ .

In the general case, for arbitrary  $n$ , all expressions as well as the condition (30) are too cumbersome therefore we present here this condition for  $n = 0$ , as an illustration

$$(\tau + \frac{\lambda}{v_z})(\frac{1}{\mathcal{E}_0} + \mathcal{E}_0 + \eta) + (\tau v + \lambda)\frac{\varepsilon}{c}(\frac{1}{\mathcal{E}_0} - \mathcal{E}_0 + \eta) = 0. \quad (33)$$

Another parameter, which may be measured, is the frequency  $\omega$ . This frequency differs from (27), and at  $n = 0$  is as follows

$$\omega = \frac{(1 + \mathcal{E}_0^2)}{2\sqrt{2}}\omega_m. \quad (34)$$

Once again the surprising fact, both frequencies coincide provided  $\mathcal{E}_0^2 = 1$ .

### CONCLUSION

Fermions in the field of a traveling, circularly polarized electromagnetic wave and a constant magnetic field are localized in the small cross-section with the size of the order of  $l_c$ . Singular solutions, i.e., solutions with energy in vicinity of the singular point are of considerable interest, from viewpoint of experiment. Such solutions arise only at certain values of the longitudinal momentum. The average energy for singular states may exceed more than twice the energy of rest mass. In contrast to the non-relativistic case, the longitudinal component of spin for all states equals zero in the first approximation.

For single states the transverse momentum and spin are oscillated similarly to the amplitude magnetic or electric field of the traveling circularly polarized wave. For mixed states temporal behavior of transverse components of spin differs from the non-relativistic case by the form of the temporal change. Average energy of such states has a minimum of the order of  $2mc^2$ . The condition of this minimum exactly coincides with the classical condition of magnetic resonance at  $g$ -factor 2. Possible, mixed states consisting of states with the same momentum, at the minimum energy, have chances to be stable.

Physical arguments in favor of the statement that the transformation, for the transition to the rotating frame, must be non-Galilean are presented in the paper.

From the above studies the conclusion follows that some parameters of this transformation can be measured not only in optics, but also in high-energy physics. Such

measurements are very important since the parameters determine limits of using physical laws on very small intervals of time  $< 10^{-23}$  sec and length  $< 10^{-13}$  cm.

- 
- [1] B. V. Gisin, arXiv: 2011.3832v5 [math-ph] 29 Sep 2011.
  - [2] L.D. Landau & E.M. Lifshitz, *Quantum mechanics*, Volume 3 of A Course of Theoretical Physics, (Pergamon Press, 1965).
  - [3] B. V. Gisin, *Optical measurement of a fundamental constant with the dimension of time*, arXiv:1307.7354v2 [physics.gen-ph] 10 Nov 2013.
  - [4] J. P. Campbell, and W. H. Steier, *Rotating-Waveplate Optical-Frequency Shifting in Lithium Niobate*, IEEE J. Quantum Electron. **QE-7**, 450-457 (1971).